Computing the Rank profile matrix (and some bonus)

Jounées Nationales du Calcul Formel

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Cluny.
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Trailer

Which CPU arithmetic to multiply 2000×2000 matrices over 200bit integers ?

1. boolean
2. `int32_t`
3. `int64_t`
4. `float`
5. `double`
6. GMP `mpz_t` (hence `uint64_t`)

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Rank profiles: how to select the first 3 linearly indep rows of

$$\begin{array}{|ccccc|}
\hline
0 & 0 & 0 & 0 & 1 \\
0 & 2 & 2 & 0 & 0 \\
0 & 1 & 1 & 1 & 2 \\
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\end{array}$$

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Outline

1. Choosing the underlying arithmetic
   - Using machine word arithmetic
   - Larger field sizes

2. Reductions and building blocks

3. Gaussian elimination
   - Which reduction
   - Computing rank profiles
   - Algorithmic instances
   - Relation to other decompositions
   - The small rank case

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Most common operation

Most of dense linear algebra operations boil down to (a lot of)

$$\texttt{y} \leftarrow \texttt{y} \pm \texttt{a} * \texttt{b}$$

- dot-product
- matrix-matrix multiplication
- rank 1 update in Gaussian elimination
- Schur complements, ...

Which computer arithmetic ?

Many base fields/rings to support

$\mathbb{Z}_2$	1 bit
$\mathbb{Z}_{3,5,7}$	2-3 bits
$\mathbb{Z}_p$	4-26 bits
$\mathbb{Z}, \mathbb{Q}$	$>$ 32 bits
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- boolean
- integer (fixed size)
- floating point
- .. and their vectorization

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Dense linear algebra over $\mathbb{Z}_p$ for word-size p

Delayed modular reductions

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2. Reduce modulo p only when necessary

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When to reduce ?

Bound the values of all intermediate computations.

- A priori:

Representation of $\mathbb{Z}_p$	$\{0 \dots p-1\}$	$\{-\frac{p-1}{2} \dots \frac{p-1}{2}\}$
Scalar product, Classic MatMul	$n(p-1)^2$	$n\left(\frac{p-1}{2}\right)^2$

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Strassen-Winograd MatMul (ℓ rec. levels)	$\left(\frac{1+3^\ell}{2}\right)^2\left\lfloor\frac{n}{2^\ell}\right\rfloor(p-1)^2$	$9^\ell\left\lfloor\frac{n}{2^\ell}\right\rfloor\left(\frac{p-1}{2}\right)^2$

Computing over fixed size integers

How to compute with (machine word size) integers efficiently?

1. use CPU's **integer arithmetic units**
   `y += a * b`: correct if $|ab + y| < 2^{63} \rightsquigarrow |a|, |b| < 2^{31}$

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addsd    %xmm0, %xmm1
movq     %xmm1, %rax
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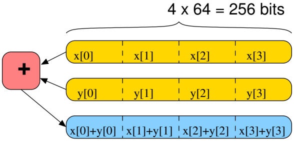
```
movsd    (%rax,%rdx,8), %xmm0      vinsertf128   $0x1, 16(%rcx,%rax), %ymm0,
mulsd    -56(%rbp), %xmm0          vmulpd        %ymm1, %ymm0, %ymm0
addsd    %xmm0, %xmm1              vaddpd        (%rsi,%rax),%ymm0, %ymm0
movq     %xmm1, %rax               vmovapd       %ymm0, (%rsi,%rax)
```

Exploiting *in-core* parallelism

Ingredients

SIMD: Single Instruction Multiple Data:
1 arith. unit acting on a vector of data

MMX	64 bits
SSE	128bits
AVX	256 bits
AVX-512	512 bits



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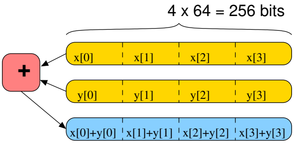
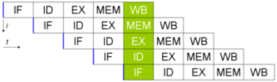


Pipeline: amortize the latency of an operation when used repeatedly

throughput of 1 op/ Cycle for all arithmetic ops considered here



IF	ID	EX	MEM	WB				
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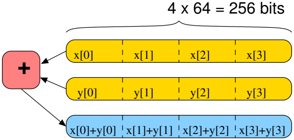

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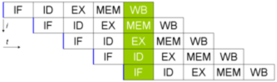

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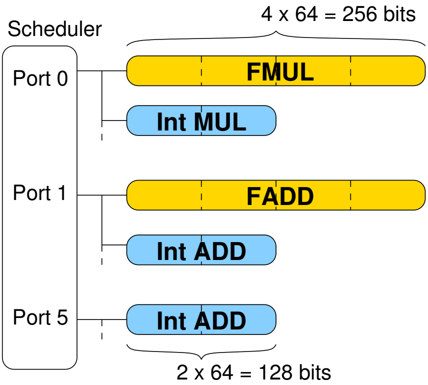
Execution Unit parallelism: multiple arith. units acting simulatneously on distinct registers

SIMD and vectorization

Intel Sandybridge micro-architecture



Performs at every clock cycle:

- 1 Floating Pt. Mul × 4
- 1 Floating Pt. Add × 4

Or:

- 1 Integer Mul × 2
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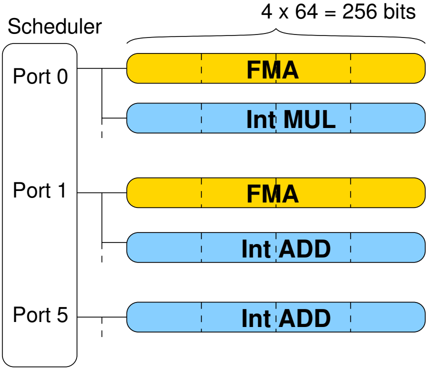
SIMD and vectorization

Intel Haswell micro-architecture



Performs at every clock cycle:

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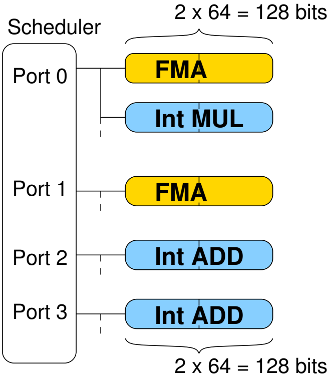
FMA: Fused Multiplying & Accumulate, `c += a * b`

SIMD and vectorization

AMD Bulldozer micro-architecture



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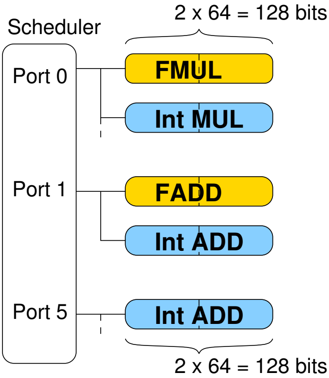

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Intel Haswell	INT	256	2	1		4	3.5	**28**	
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Computing over fixed size integers: ressources

Micro-architecture bible: Agner Fog's software optimization resources
[`www.agner.org/optimize`]

Experiments:

`dgemm (double)`: `OpenBLAS` [`http://www.openblas.net/`]

`igemm (int64_t)`: `Eigen` [`http://eigen.tuxfamily.org/`] &
`FFLAS-FFPACK` [`linalg.org/projects/fflas-ffpack`]

Looking into the near future

Intel Skylake & Knights Landing: AVX512-F

2016 (2017 on Xeons)

- Enlarge SIMD register width to 512 bits (8 `double` or `int64_t`)
- same micro arch : FMA for FP and seprate mul/add for INT.

Looking into the near future

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Cannonlake: AVX512-IFMA

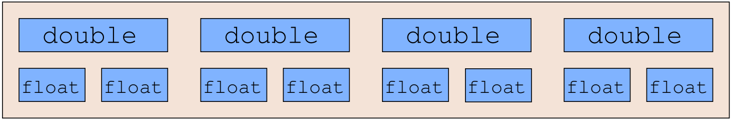
$>$2017

- AVX512 extension: IFMA (Integer FMA): `y += a*b` on `int64_t`
- But limited to the lower 52 bits of the output (uses the FP FMA)

$\rightsquigarrow$ no advantage for `int64_t` over `double`

Integer Packing

32 bits: half the precision twice the speed

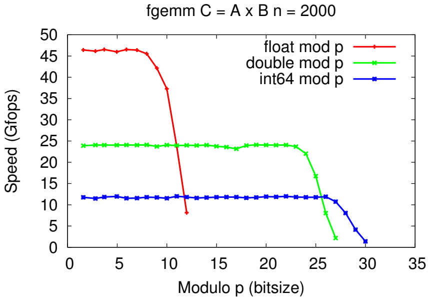


double double double double

float float float float float float float float

Gfops	`double`	`float`	`int64_t`	`int32_t`
Intel SandyBridge	24.7	49.1	12.1	24.7
Intel Haswell	49.2	77.6	23.3	27.4
AMD Bulldozer	13.0	20.7	6.63	11.8

Computing over fixed size integers

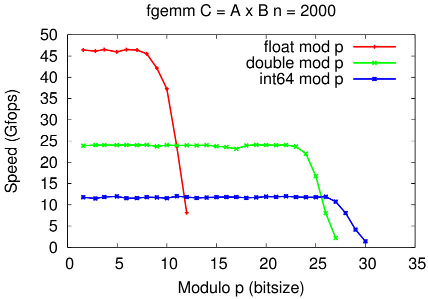


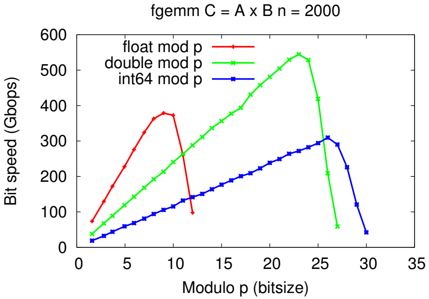
SandyBridge i5-3320M@3.3Ghz. $n = 2000$.

Take home message

- Floating pt. arith. delivers the highest speed (except in corner cases)
- 32 bits twice as fast as 64 bits

Computing over fixed size integers





SandyBridge i5-3320M@3.3Ghz. $n = 2000$.

Take home message

- Floating pt. arith. delivers the highest speed (except in corner cases)
- 32 bits twice as fast as 64 bits
- best bit computation throughput for double precision floating points.

Larger finite fields: $\log_2 p \geq 32$

As before:

1. Use adequate integer arithmetic
2. reduce modulo p only when necessary

Which integer arithmetic?

1. big integers (GMP)
2. fixed size multiprecision (Givaro-RecInt)
3. Residue Number Systems (Chinese Remainder theorem)
   $\rightsquigarrow$ using moduli delivering optimum bitspeed

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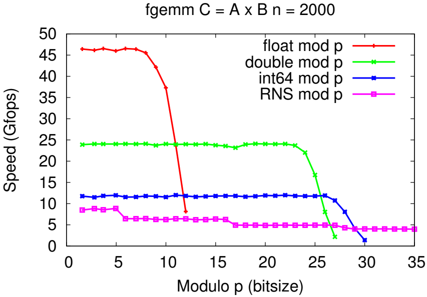
$\log_2 p$	50	100	150
GMP	58.1s	94.6s	140s
RecInt	5.7s	28.6s	837s
RNS	0.785s	1.42s	1.88s

$n = 1000$, on an Intel SandyBridge.

In practice


fgemm C = A x B n = 2000
Speed (Gfops)
Modulo p (bitsize)
float mod p
double mod p
int64 mod p
RNS mod p
0
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50
0
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25
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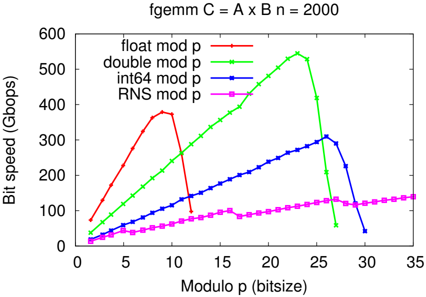
35


fgemm C = A x B n = 2000
Bit speed (Gbops)
Modulo p (bitsize)
float mod p
double mod p
int64 mod p
RNS mod p
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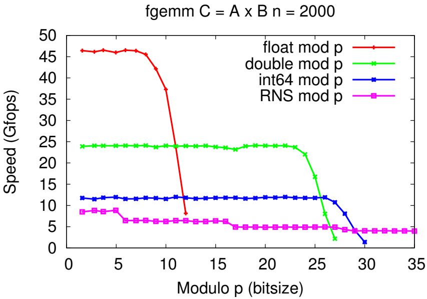
35

In practice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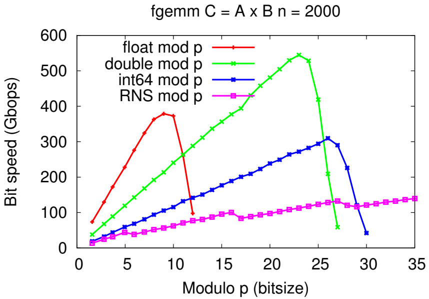
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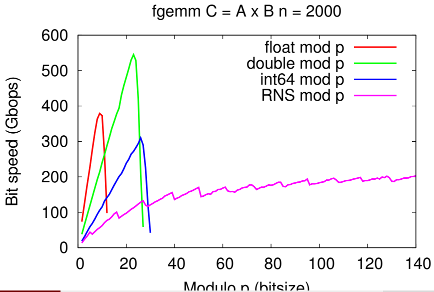
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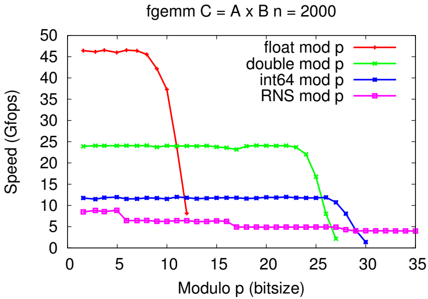
In practice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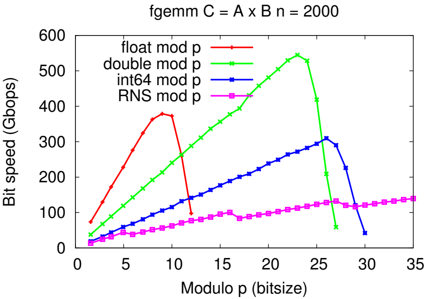
fgemm C = A x B n = 2000
Bit speed (Gbops)
Modulo p (bitsize)
float mod p
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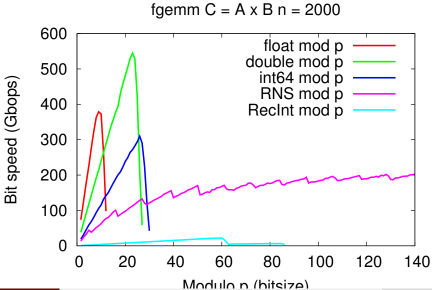
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Outline

1. Choosing the underlying arithmetic
   - Using machine word arithmetic
   - Larger field sizes
2. Reductions and building blocks
3. Gaussian elimination
   - Which reduction
   - Computing rank profiles
   - Algorithmic instances
   - Relation to other decompositions
   - The small rank case

Reductions to building blocks

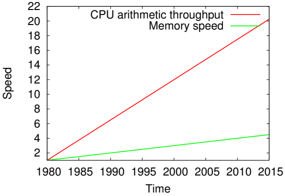
Huge number of algorithmic variants for a given computation.

$\rightsquigarrow$ Need to structure the design for a set of routines :

- Focus tuning effort on a single routine
- Some operations deliver better efficiency:
  - in practice: memory access pattern
  - in theory: better algorithms

Memory access pattern

- **The memory wall:** communication speed improves slower than arithmetic



Memory access pattern

- **The memory wall:** communication speed improves slower than arithmetic
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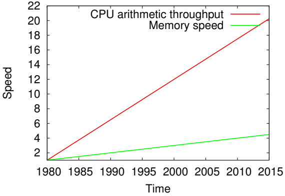

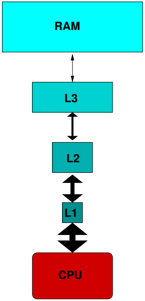

Memory access pattern

- **The memory wall:** communication speed improves slower than arithmetic
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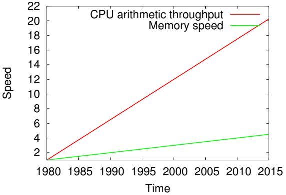
⇝ Need to overlap communications by computation

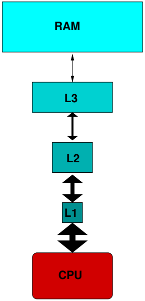
Design of BLAS 3 [Dongarra & Al. 87]

- Group all ops in Matrix products `gemm`:

  Work $O(n^3) >> $ Data $O(n^2)$

MatMul has become a building block in practice





Sub-cubic linear algebra

< 1969: $O(n^3)$ for everyone (Gauss, Householder, Danilevskĭi, etc)

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Matrix Multiplication $\rightsquigarrow O(n^\omega)$

[Strassen 69]:	$O(n^{2.807})$
⋮	
[Schönhage 81]	$O(n^{2.52})$
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[Coppersmith, Winograd 90]	$O(n^{2.375})$
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[Schönhage 72]:	QR in $O(n^\omega)$
[Bunch, Hopcroft 74]:	LU in $O(n^\omega)$
[Ibarra & *al.* 82]:	Rank in $O(n^\omega)$
[Keller-Gehrig 85]:	CharPoly in $O(n^\omega \log n)$

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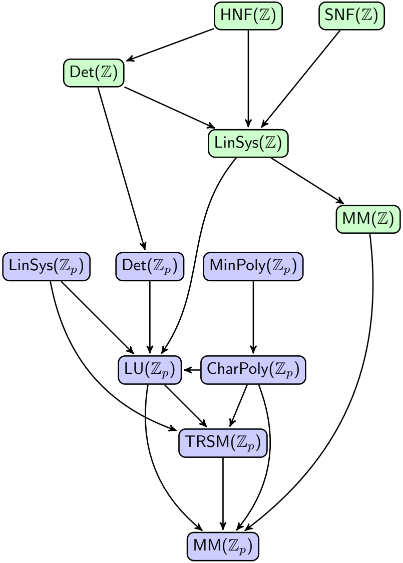
MatMul has become a building block in theoretical reductions

Reductions: theory



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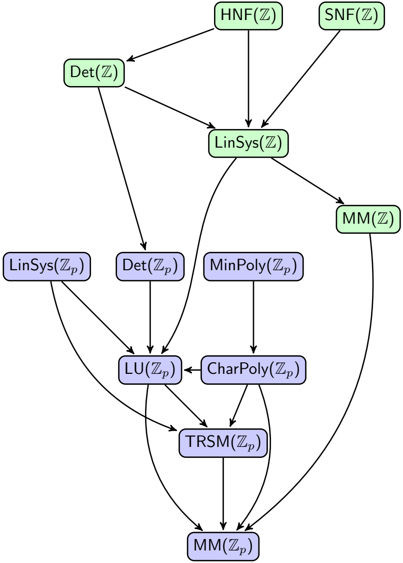
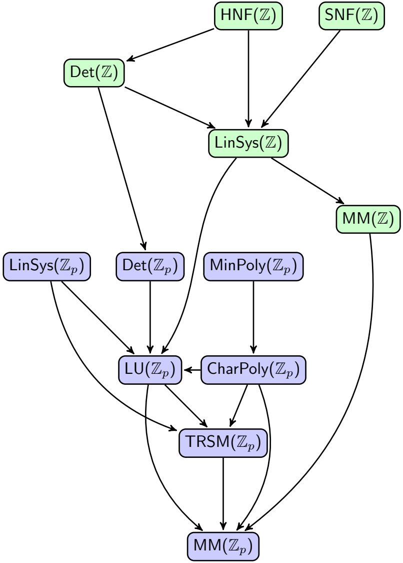


Common mistrust

Fast linear algebra is

- ✗ never faster
- ✗ numerically unstable

Reductions: theory and practice



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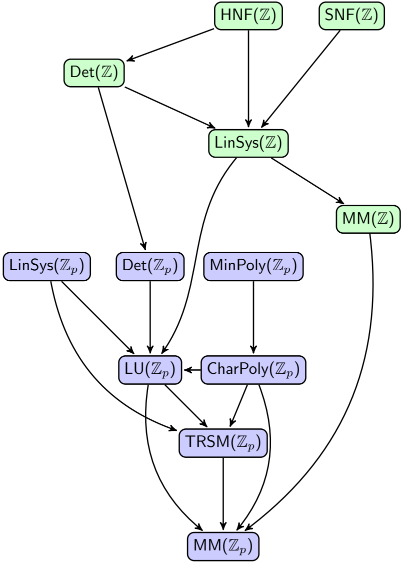
- ✓ same building block **in theory** and **in practice**

⇝ reduction trees are still relevant

Reductions: theory and practice



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Road map towards efficiency in practice

1. Tune the MatMul building block.
2. Tune the reductions.

Putting it together: MatMul building block over $\mathbb{Z}/p\mathbb{Z}$

Ingedients [FFLAS-FFPACK library]

- Compute over $\mathbb{Z}$ and delay modular reductions

$$\rightsquigarrow \boxed{k\left(\frac{p-1}{2}\right)^2 < 2^{\text{mantissa}}}$$

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- Fastest integer arithmetic: `double`
- Cache optimizations

$\rightsquigarrow$ numerical BLAS

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$\rightsquigarrow$ $9^\ell \left\lfloor \frac{k}{2^\ell} \right\rfloor \left(\frac{p-1}{2}\right)^2 < 2^{53}$

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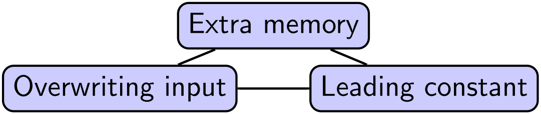
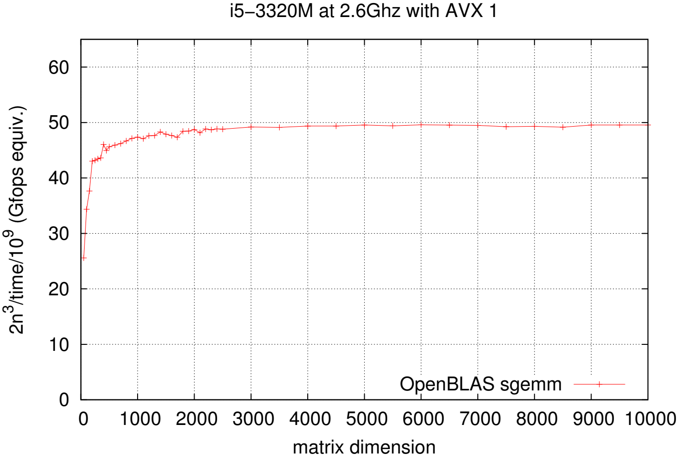
with memory efficient schedules [Boyer, Dumas, P. and Zhou 09]
Tradeoffs:

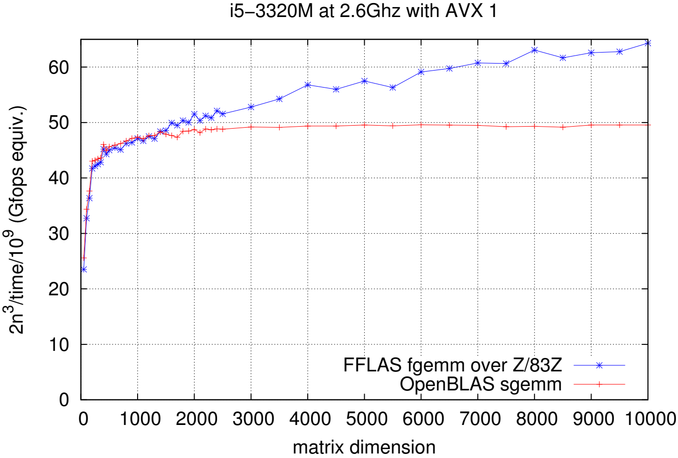


Fully in-place in
$7.2n^{2.807} + \ldots$

Sequential Matrix Multiplication

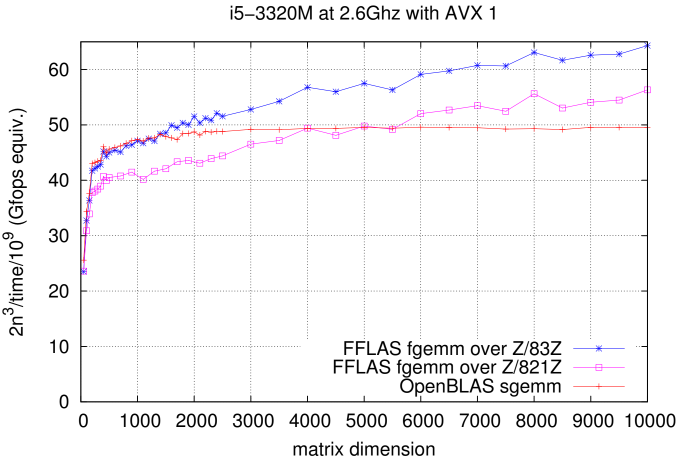


Sequential Matrix Multiplication



$p = 83$, $\rightsquigarrow$ 1 mod / 10000 mul.

Sequential Matrix Multiplication



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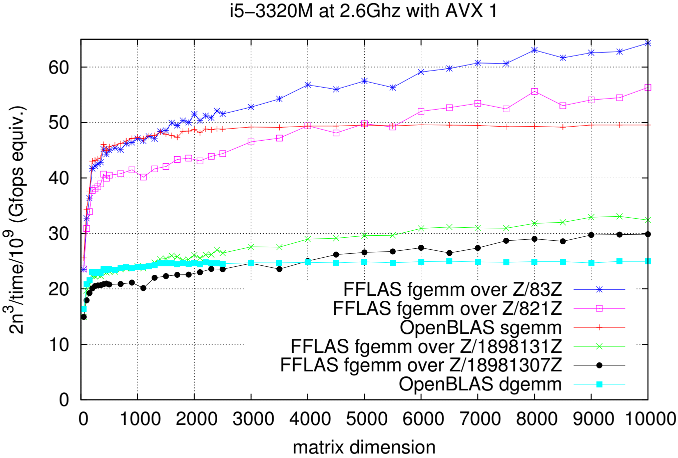
$p = 821$, $\rightsquigarrow$ 1 mod / 100 mul.

Sequential Matrix Multiplication



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$p = 1898131$, ⇝ 1 mod / 10000 mul.

$p = 18981307$, ⇝ 1 mod / 100 mul.

Reductions in dense linear algebra

LU decomposition

- Block recursive algorithm $\rightsquigarrow$ reduces to MatMul $\rightsquigarrow$ $O(n^\omega)$

n	1000	5000	10000	15000	20000
`LAPACK-dgetrf`	**0.024s**	**2.01s**	**14.88s**	48.78s	113.66
`fflas-ffpack`	0.058s	2.46s	16.08s	**47.47s**	**105.96s**

Intel Haswell E3-1270 3.0Ghz using OpenBLAS-0.2.9

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Characteristic Polynomial

- A new reduction to matrix multiplication in $O(n^\omega)$.

n	1000	2000	5000	10000
`magma-v2.19-9`	1.38s	24.28s	332.7s	2497s
`fflas-ffpack`	**0.532s**	**2.936s**	**32.71s**	**219.2s**

Intel Ivy-Bridge i5-3320 2.6Ghz using OpenBLAS-0.2.9

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Outline

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   - Computing rank profiles
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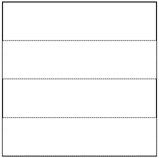
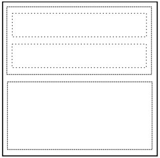
The case of Gaussian elimination

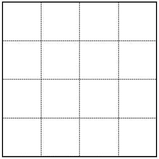
Which reduction to MatMul ?



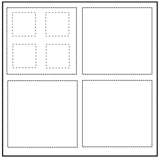
Slab iterative
`LAPACK`



Slab recursive
`FFLAS-FFPACK`



Tile iterative
`PLASMA`



Tile recursive
`FFLAS-FFPACK`

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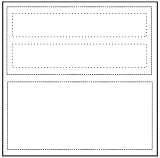
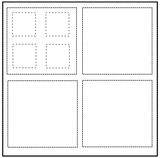
Which reduction to MatMul ?



Slab recursive

FFLAS-FFPACK



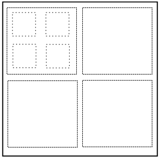
Tile recursive

FFLAS-FFPACK

- Sub-cubic complexity: recursive algorithms

The case of Gaussian elimination

Which reduction to MatMul ?



Tile recursive

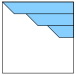
FFLAS-FFPACK

- Sub-cubic complexity: recursive algorithms
- Data locality

Computing rank profiles

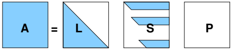
Rank profiles: first linearly independent columns

- Major invariant of a matrix (echelon form)
- Gröbner basis computations (Macaulay matrix)
- Krylov methods

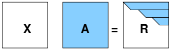


Gaussian elimination revealing echelon forms:

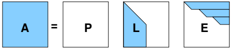
[Ibarra, Moran and Hui 82]



[Keller-Gehrig 85]



[Jeannerod, P. and Storjohann 13]



Computing rank profiles

Lessons learned (or what we thought was necessary):

- treat rows in order
- exhaust all columns before considering the next row
- **slab** block splitting required (recursive or iterative)
  $\rightsquigarrow$ similar to partial pivoting

Need for a more flexible pivoting

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Gathering linear independence invariants

Two ways to look at a matrix: row- or column-wise

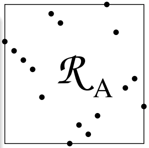
- Row rank profile, column echelon form,
- Column rank profile, row echelon form,

Can't a unique invariant capture all information ?

The rank profile matrix [Dumas, P. and Sultan'15]

Definition (Rank Profile matrix)

The unique $\mathcal{R}_A \in \{0,1\}^{m\times n}$ such that any pair of (i,j)-leading sub-matrix of $\mathcal{R}_A$ and of A have the same rank.



$$\overset{\mathrm{A}}{\boxed{\begin{matrix}1 & 2 & 3 & 4\\ 2 & 4 & 5 & 8\\ 1 & 2 & 3 & 4\\ 3 & 5 & 9 & 12\end{matrix}}} \longrightarrow \overset{\boldsymbol{\mathcal{R}}}{\boxed{\begin{matrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{matrix}}}$$

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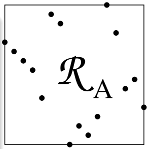
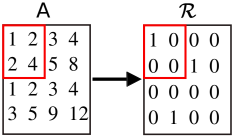


Theorem

- *RowRP and ColRP read directly on* $\mathcal{R}(A)$
- *Same holds for any* (i,j)*-leading submatrix.*



$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 8 \\ 1 & 2 & 3 & 4 \\ 3 & 5 & 9 & 12 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

RowRP $= \{1\}$
ColRP $= \{1\}$

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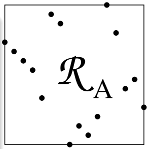

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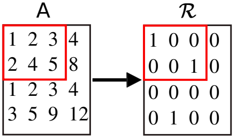

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 8 \\ 1 & 2 & 3 & 4 \\ 3 & 5 & 9 & 12 \end{bmatrix} \longrightarrow \mathcal{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

RowRP = {1,2}
ColRP = {1,3}

The rank profile matrix [Dumas, P. and Sultan'15]

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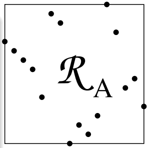

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RowRP $=$ {1,4}
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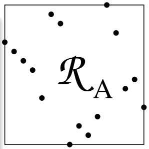

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RowRP = {1,4}
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$$A = PLUQ = P\begin{bmatrix} L & 0 \\ M & I_{m-r} \end{bmatrix} \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} \begin{bmatrix} U & V \\ & I_{n-r} \end{bmatrix} Q$$

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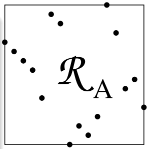

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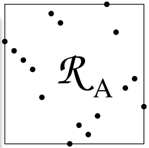

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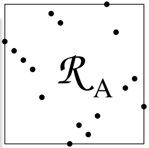

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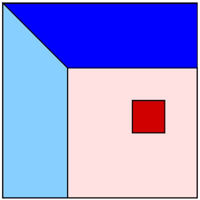
When does $\boxed{\Pi_{P,Q} = \mathcal{R}(A)}$?

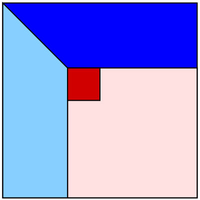
Anatomy of a PLUQ decomposition



Four types of elementary operations:

Search: finding a pivot

Anatomy of a PLUQ decomposition



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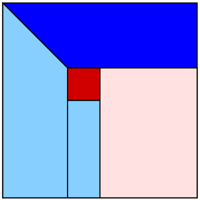
Anatomy of a PLUQ decomposition



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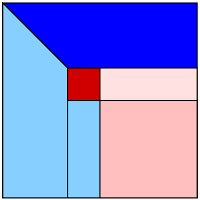

Four types of elementary operations:

Search: finding a pivot

Permutation: moving the pivot to the main diagonal

Normalization: computing L: $l_{i,k} \leftarrow \frac{a_{i,k}}{a_{k,k}}$

Update: applying the elimination $a_{i,j} \leftarrow a_{i,j} - \frac{a_{i,k}a_{k,j}}{a_{k,k}}$

Impact on the PLUQ decomposition

Normalization: determines whether L or U is unit diagonal

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Problem (Reformulation)

Under what conditions on the **Search** *and* **Permutation** *operations does a PLUQ decomposition algorithm reveals RowRP, ColRP or $\mathcal{R}_A$?*

The Pivoting matrix

Definition (The pivoting matrix)

Given a PLUQ decomposition $A = PLUQ$ with rank r, define

$$\Pi_{P,Q} = P \begin{bmatrix} I_r & \\ & \end{bmatrix} Q.$$

Locates the position of the pivots in the matrix A.

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Problem (Rank profile revealing PLUQ decompositions)

Under which conditions

- $\Pi_{P,Q} = \mathcal{R}_A$

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Under which conditions

- $\Pi_{P,Q} = \mathcal{R}_A$
- $RowSupp(\Pi_{P,Q}) = RowSupp(\mathcal{R}_A) = RowRP(A)$ *(Weaker)*
- $ColSupp(\Pi_{P,Q}) = ColSupp(\mathcal{R}_A) = ColRP(A)$ *(Weaker)*

The Search operation

Various strategies depending on the context

Numerical stability: find the absolute largest pivot

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Search revealing rank profiles

- No stability issue over exact domains
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Example

Search: "Any non zero element on the topmost row":

$$A = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \Rightarrow \mathcal{R}_A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

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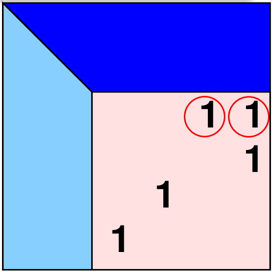
Pivoting and permutation strategies

Pivot Search

Pivot's (i,j) position minimizes some pre-order:

Row order: any non-zero on the first non-zero row



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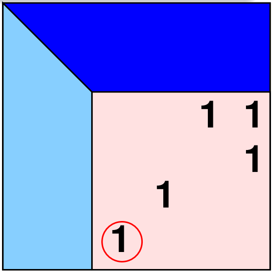


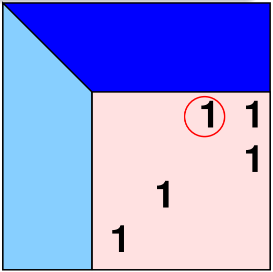
Pivoting and permutation strategies

Pivot Search

Pivot's (i, j) position minimizes some pre-order:

Row/Col order: any non-zero on the first non-zero row/col

Lex order: first non-zero on the first non-zero row



Pivoting and permutation strategies

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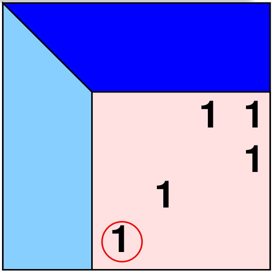
Lex/RevLex order: first non-zero on the first non-zero row/col



Pivoting and permutation strategies

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Pivot's (i, j) position minimizes some pre-order:

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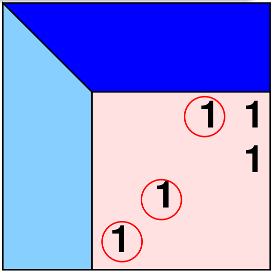
Lex/RevLex order: first non-zero on the first non-zero row/col

Product order: first non-zero in the (i, j) leading sub-matrix



Sufficient ?

Is lexicographic ordering sufficient to reveal both rank profiles?

Example

With a lexicographic ordering

1. $A = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \Rightarrow \mathcal{R}_A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \Pi_{P,Q}$

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$\rightsquigarrow$ Pivot Swaps mix-up precedence between rows/cols.

$\rightsquigarrow$ **Permutations** also have to be considered

Pivoting and permutation strategies

Pivot Search

Pivot's (i,j) position minimizes some pre-order:

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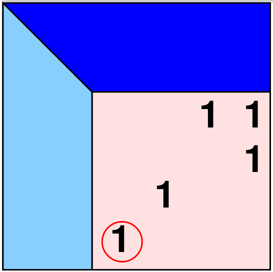
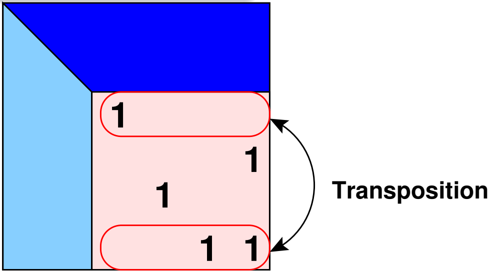
Permutation

- Transpositions





Pivoting and permutation strategies

Pivot Search

Pivot's (i, j) position minimizes some pre-order:

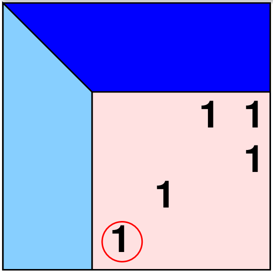
Row/Col order: any non-zero on the first non-zero row/col

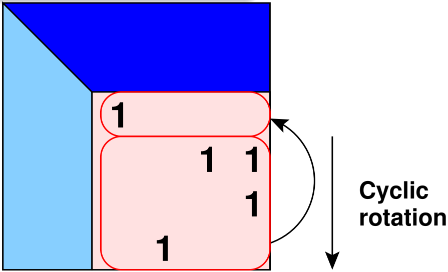
Lex/RevLex order: first non-zero on the first non-zero row/col

Product order: first non-zero in the (i, j) leading sub-matrix

Permutation

- Transpositions
- Cyclic Rotations





Pivoting strategies revealing rank profiles

For any type of PLUQ algorithm: iterative / block iterative / recursive

Search	**Row perm.**	**Col. perm.**	RowRP	ColRP	$\mathcal{R}_A$	**Instance**
Row order Col. order						
Lexico.						
Rev. lex.						
Product						

Pivoting strategies revealing rank profiles

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Row order Col. order	Transposition	Transposition	✓			[IMH82] [JPS13]
Lexico.						
Rev. lex.						
Product						

- RowRP $= \begin{bmatrix}1 & 2 & \dots & m\end{bmatrix} P \begin{bmatrix} I_r \\ 0 \end{bmatrix}$

Pivoting strategies revealing rank profiles

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Search	**Row perm.**	**Col. perm.**	RowRP	ColRP	$\mathcal{R}_A$	**Instance**
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Row order	Transposition	Transposition	✓			[IMH82] [JPS13]
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Lexico.	Transposition	Transposition	✓			[Sto00]
Rev. lex.	Transposition	Transposition		✓		[Sto00]
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Lexico.	Transposition	Transposition	✓			[Sto00]
Rev. lex.	Transposition	Transposition		✓		[Sto00]
Product	Rotation	Rotation	✓	✓	✓	[DPS13]

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Pivoting strategies revealing rank profiles

For any type of PLUQ algorithm: iterative / block iterative / recursive

Search	**Row perm.**	**Col. perm.**	RowRP	ColRP	$\mathcal{R}_A$	**Instance**
Row order	Transposition	Transposition	✓			[IMH82] [JPS13]
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Rev. lex.	Transposition	Transposition		✓		[Sto00]
Product	Rotation	Transposition	✓			[DPS15]
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Pivoting strategies revealing rank profiles

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Lexico.	Transposition	Transposition	✓			[Sto00]
Lexico.	Transposition	Rotation	✓	✓	✓	[DPS15]
Lexico.	Rotation	Rotation	✓	✓	✓	[DPS15]
Rev. lex.	Transposition	Transposition		✓		[Sto00]
Rev. lex.	Rotation	Transposition	✓	✓	✓	[DPS15]
Rev. lex.	Rotation	Rotation	✓	✓	✓	[DPS15]
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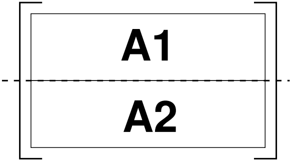
The slab recursive algorithm

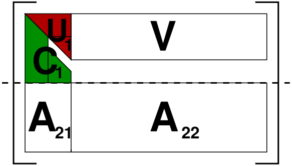
Slab Recursive LU [IMH82, KG85, Sto00, JPS13]



1. Split A Row-wise

The slab recursive algorithm

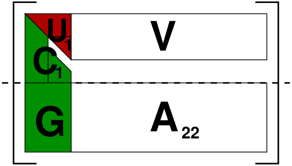
Slab Recursive LU [IMH82, KG85, Sto00, JPS13]



1. Split A Row-wise
2. Recursive call on A_1

The slab recursive algorithm

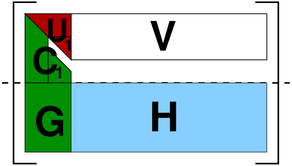
Slab Recursive LU [IMH82, KG85, Sto00, JPS13]



1. Split A Row-wise
2. Recursive call on A_1
3. $G \leftarrow A_{21}U_1^{-1}$ (`trsm`)

The slab recursive algorithm

Slab Recursive LU [IMH82, KG85, Sto00, JPS13]



1. Split A Row-wise
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4. $H \leftarrow A_{22} - G \times V$ (`MM`)

The slab recursive algorithm

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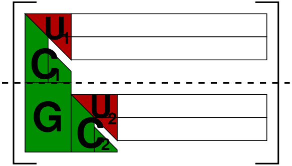

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The slab recursive algorithm

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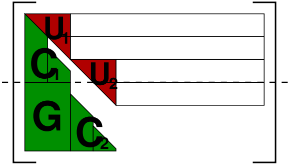

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6. Row permutations

The slab recursive algorithm

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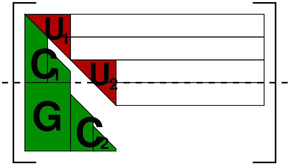

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Implements the lexicographic order search.

- Col/Row Transpositions : Computes the ColRP

The slab recursive algorithm

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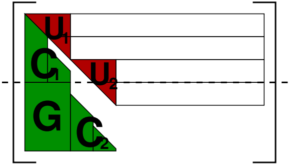

1. Split A Row-wise
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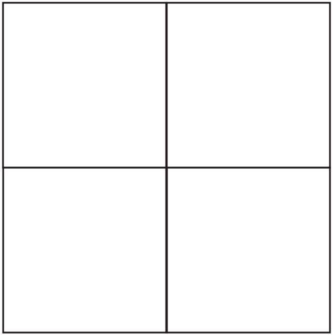
3. $G \leftarrow A_{21}U_1^{-1}$ (trsm)
4. $H \leftarrow A_{22} - G \times V$ (MM)
5. Recursive call on H
6. Row permutations

Implements the lexicographic order search.

- Col/Row Transpositions : Computes the ColRP
- Row Rotations : Computes $\mathcal{R}_A$ [DPS15]

The tiled recursive algorithm

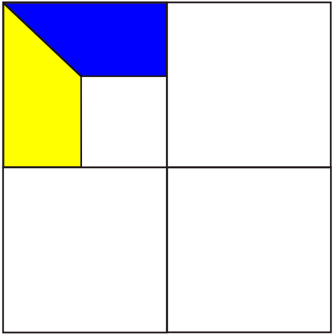
Dumas, P. and Sultan 13



2×2 block splitting

The tiled recursive algorithm

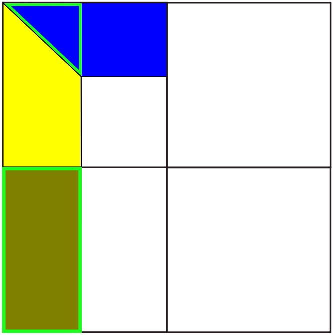
Dumas, P. and Sultan 13



Recursive call

The tiled recursive algorithm

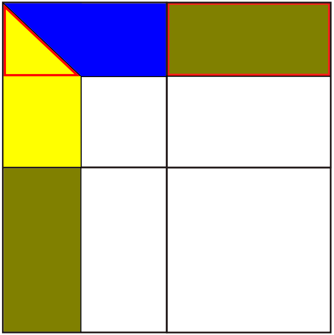
Dumas, P. and Sultan 13



TRSM: $B \leftarrow BU^{-1}$

The tiled recursive algorithm

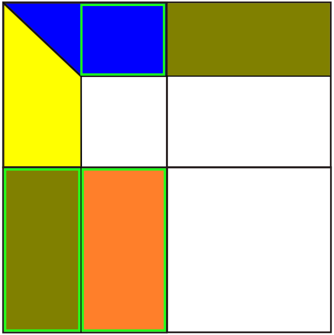
Dumas, P. and Sultan 13



TRSM: $B \leftarrow L^{-1}B$

The tiled recursive algorithm

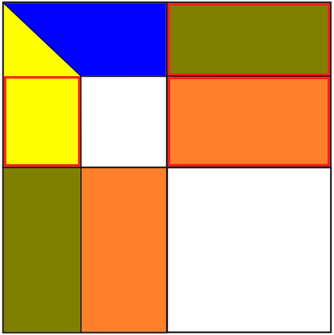
Dumas, P. and Sultan 13



MatMul: $C \leftarrow C - A \times B$

The tiled recursive algorithm

Dumas, P. and Sultan 13



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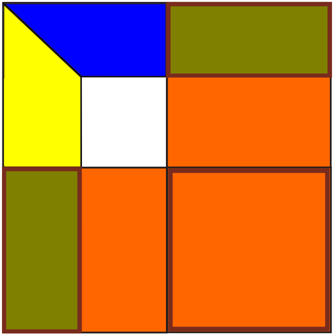

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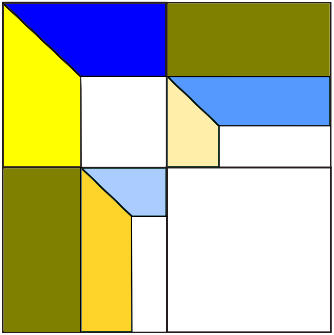
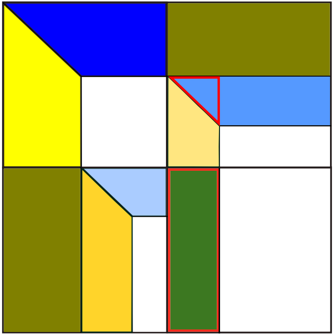


2 independent recursive calls (compatible with the **product order**)

The tiled recursive algorithm

Dumas, P. and Sultan 13



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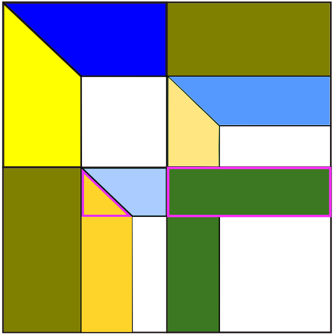
The tiled recursive algorithm

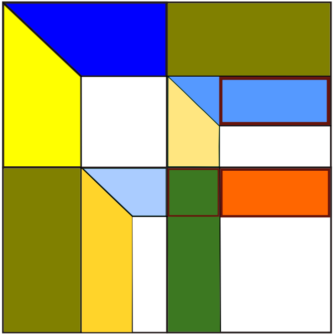
Dumas, P. and Sultan 13



TRSM: $B \leftarrow L^{-1}B$

The tiled recursive algorithm

Dumas, P. and Sultan 13



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The tiled recursive algorithm

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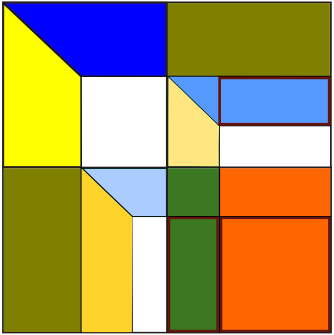

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The tiled recursive algorithm

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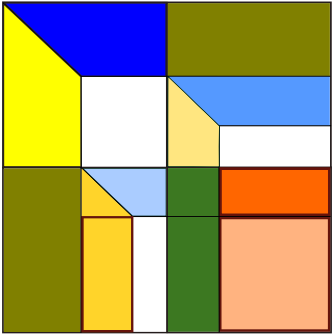

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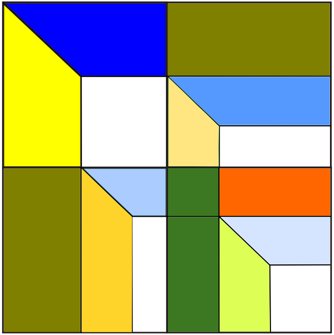
The tiled recursive algorithm

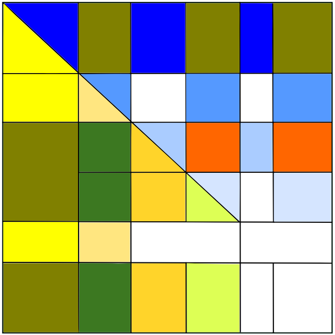
📄 Dumas, P. and Sultan 13



Recursive call

The tiled recursive algorithm

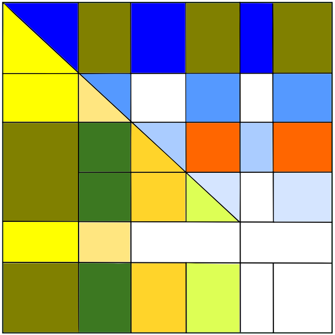
Dumas, P. and Sultan 13



Puzzle game (block **rotations**)

The tiled recursive algorithm

Dumas, P. and Sultan 13



- $O(mnr^{\omega-2})$ ($2/3n^3$ for $\omega = 3$)
- fewer modular reductions than slab algorithms
- rank deficiency introduces parallelism

Iterative algorithms

- Unefficient with large problems
- Good for base case implementations (faster in-cache computation)

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Which base case algorithm?

- Formerly [DPS13]: **product order** iterative algorithm
  - ✗ many permutations
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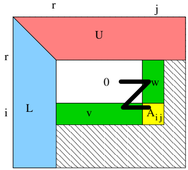

Iterative algorithms

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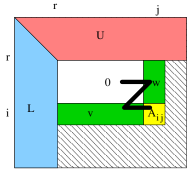
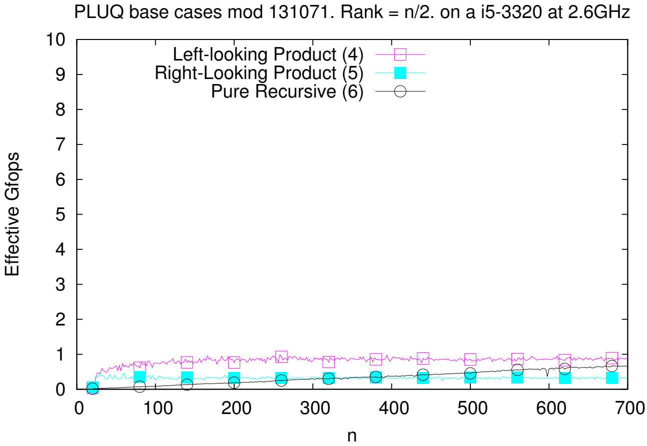
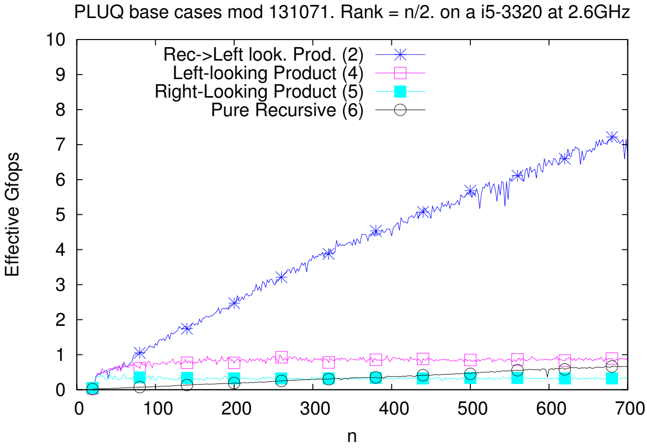


- [DPS15]: Simply use the schoolbook algorithm (Lexico+Rotations)
  - ✓ fewer permutations
  - ✓ modular reductions delayed more easily
  - ✓ Crout variant: better data access pattern


PLUQ base cases mod 131071. Rank = n/2. on a i5-3320 at 2.6GHz
Effective Gfops
0
1
2
3
4
5
6
7
8
9
10
n
0
100
200
300
400
500
600
700
Left-looking Product (4)
Right-Looking Product (5)
Pure Recursive (6)


PLUQ base cases mod 131071. Rank = n/2. on a i5-3320 at 2.6GHz
Effective Gfops
n
Rec->Left look. Prod. (2)
Left-looking Product (4)
Right-Looking Product (5)
Pure Recursive (6)
0
1
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3
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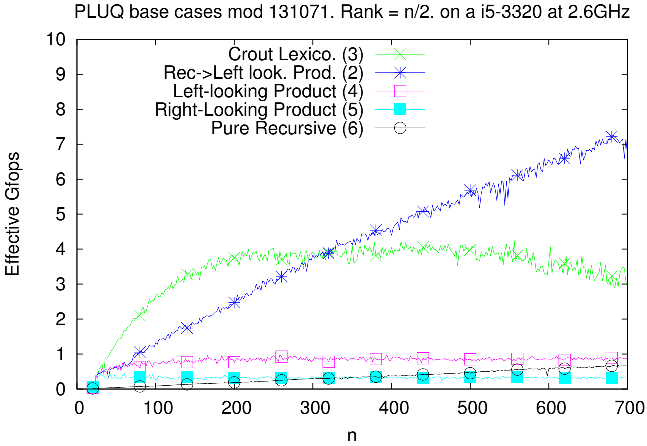
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n
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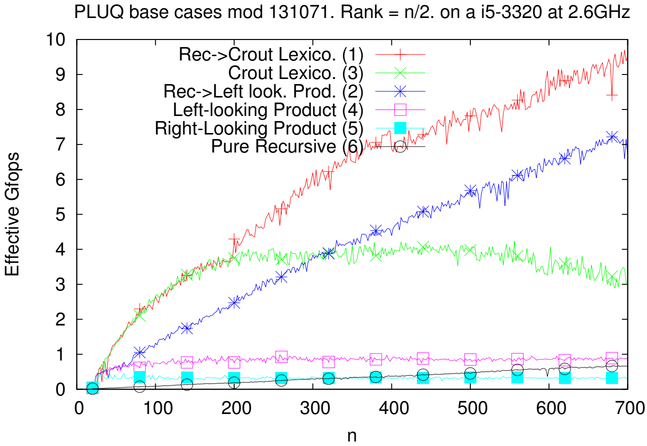
Crout Lexico. (3)
Rec->Left look. Prod. (2)
Left-looking Product (4)
Right-Looking Product (5)
Pure Recursive (6)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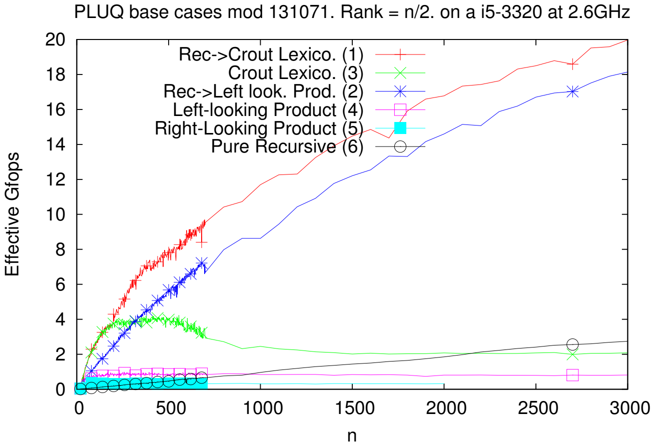
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Effective Gfops
n
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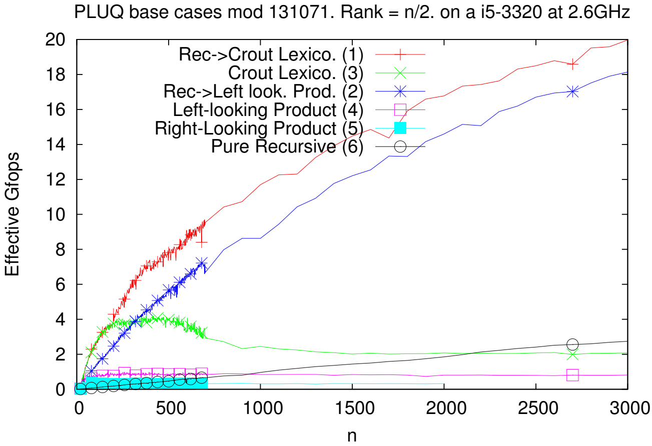
1000
1500
2000
2500
3000



- > 2 Gfops improvement
- Implemented in `FFLAS-FFPACK` (kernel of `LinBox`).

LUP and PLU decompositions

LUP

If A has generic RowRP

- $LUP(A)$ with Lex order and col. rot.: $\rightsquigarrow \begin{matrix} I_r & \\ & 0 \end{matrix} P = \mathcal{R}_A$

In particular, if A has full row rank and $m = n$: $\rightsquigarrow P = \mathcal{R}_A$

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In particular, if A has full row rank and $m = n$: $\rightsquigarrow P = \mathcal{R}_A$

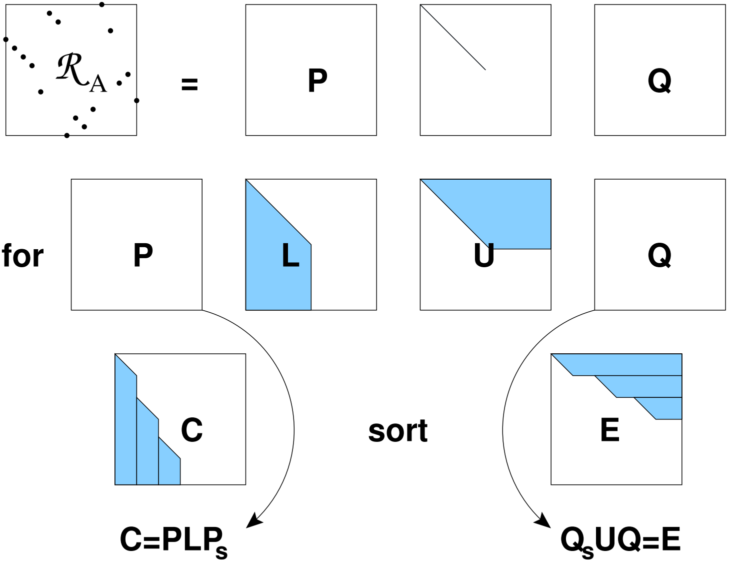
PLU

If A has generic ColRP

- $PLU(A)$ with RevLex order and row rot. $\rightsquigarrow P \begin{matrix} I_r & \\ & 0 \end{matrix} = \mathcal{R}_A$

In particular, if A has full column rank and $m = n$: $\rightsquigarrow P = \mathcal{R}_A$

Echelon forms



Small rank

When $r \ll m, n$, $O(mnr^{\omega-2})$ can be too expensive.
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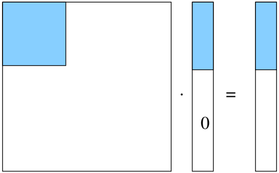
Can the rank profile matrix be computed in similar complexities?

[Storjohann Yang'14] Linear System Oracle

Sketch of the $\tilde{O}(r^3 + mn)$ algorithm

Incrementally for $s = 1..\text{rank}(A)$, maintain

- an $s \times s$ invertible sub-matrix A_s of A.
- its inverse A_s^{-1}
- a partial solution $A_s x_s = b_s$ to a linear system $Ax = b$.



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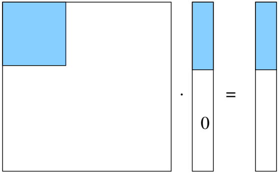

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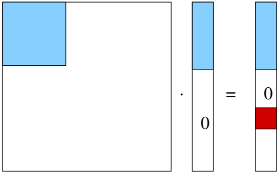

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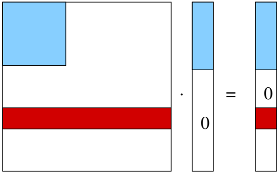

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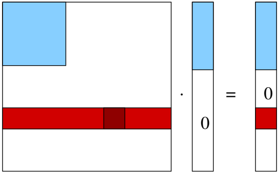

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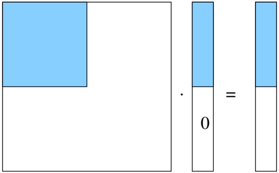

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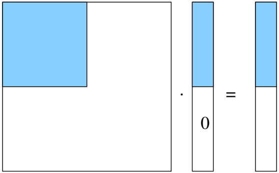

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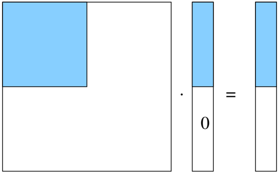

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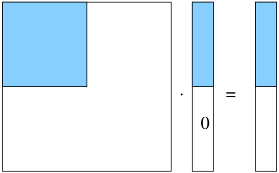

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Lexico. search with rotations ⇝ computes $\mathcal{R}_A$

[Storjohann Yang'15] Relaxed matrix inverse

Sketch of the algorithm: RowRP in $O\tilde{}(r^\omega + mn)$

1. Insted of building A_s^{-1} iteratively ($O(r^3)$), use an asymptotically fast relaxation scheme $O(r^\omega)$.
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Asymptotic reduction $>$ algorithm tuning $>$ building block implementation

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Thank you